

# Sufficient conditions on observability grammian for synchronization in arrays of coupled time-varying linear systems

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## Abstract

Synchronizability of stable, output-coupled, identical, time-varying linear systems is studied. It is shown that if the observability grammian satisfies a persistence of excitation condition, then there exists a bounded, time-varying linear feedback law that yields exponential synchronization for all fixed, asymmetrical interconnections with connected graphs. Also, a weaker condition on the grammian is given for asymptotic synchronization. No assumption is made on the strength of coupling. Moreover, related to the main problem, a particular array of output-coupled systems that is pertinent to much-studied consensus problems is investigated. In this array, the individual systems are integrators with identical, time-varying, symmetric positive semi-definite output matrices. Trajectories of this array are shown to stay bounded using a time-invariant, quadratic Lyapunov function. Also, sufficient conditions on output matrix for synchronization are provided. All of the results in the paper are generated for both continuous time and discrete time.

## 1 Introduction

When do the trajectories of a number of coupled individual systems converge to each other? This question outlines the multifaceted problem of *synchronization stability*. Unknotting this problem requires understanding the interplay of two pieces: the set of individual systems' dynamics and the (varying) topology of their coupling, i.e. who influences whom and how strongly. The general problem is insuperably difficult, which has led people to a number of simplifications, justifiable for certain applications. For instance, when the individual system dynamics is taken to be an integrator, by using convexity arguments, trajectories have been shown to converge to a fixed point in space as long as the (directed, time-varying) interconnection satisfies a fairly weak connectedness condition. Since, once synchronized, the righthand sides of the systems vanish, the word

*consensus* is used when referring to this case; see, for instance, [9, 11, 1, 7]. Another direction of investigation is fueled by the fact that the speed/occurrence of synchronization is related to the coupling (strength) between the individual systems. Studies concentrated on understanding this relation have been fruitful and significant results have emerged. We now know that the spectrum of the interconnection matrix is where we have to look at if we want to measure the strength of coupling in order to determine whether synchronization will take place or not. Roughly speaking, under the assumption that some Lyapunov function (related to the individual system dynamics only) exists, one can guarantee stability of synchronization if the coupling strength is larger than some threshold; see, for instance, [19, 10, 18, 2]. There are numerous other interesting research directions accommodating notable works in synchronization stability. We refer the interested reader to the surveys [14, 17], [4, Sec. 5].

A fundamental case in synchronization stability concerns with output-coupled identical linear systems under fixed interconnection. The problem is considered in [15] for time-invariant discrete-time systems and in [16] for continuous-time systems (as a generalization of Luenberger observer) leading to the following result: “If an individual system is detectable from its output and its system matrix is neutrally stable, then there exists a linear feedback law under which the trajectories of the coupled replicas of the individual system exponentially synchronize provided that the (directed) graph representing the interconnection is connected.” We emphasize that (i) the result needs no assumption on the strength of coupling and (ii) synchronizing feedback law is independent of the number of systems and their interconnection. In this paper we extend this result for time-varying linear systems.

For a time-varying pair  $(C, A)$ , where  $A(\cdot)$  is the system matrix and  $C(\cdot)$  is the output matrix, we first define *synchronizability* (with respect to set of all connected interconnections.) Roughly, a pair  $(C, A)$  is synchronizable if one can find a bounded time-varying linear feedback law  $L(\cdot)$  under which the trajectories of the coupled replicas of the individual system described by triple  $(C, A, L)$  synchronize for all connected interconnections. Then we study the conditions that would imply synchronizability. The assumptions and results almost parallel the time-invariant case. The assumption we make on the system matrix is that its state transition matrix is bounded in both forward and backward time<sup>1</sup>, which yields (considering trajectories) sustained and bounded oscillations. Boundedness in forward time is necessary for stability because we make no assumption on the strength of coupling. Boundedness in backward time can be relaxed at the expense of complicity of analysis and need for additional technical assumptions on pair  $(C, A)$ . For simplicity, therefore, we choose to keep it. One of the contributions of this work are in establishing the following results:

- If pair  $(C, A)$  is asymptotically observable then it is synchronizable.

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<sup>1</sup>Bounded state transition matrix assumption can be encountered in seemingly different problems in the literature whenever observability is at stake; see, for instance, [5].

- If pair  $(C, A)$  is uniformly observable then it is exponentially synchronizable.<sup>2</sup>

*Asymptotic observability* we define as that the integrand of the observability grammian satisfies a general (yet technical) condition. This condition, which we name *sufficiency of excitation*<sup>3</sup>, is significantly weaker than persistence of excitation and allows the following result, cf. [13, Thm. 2.5.1].

- Let  $Q$  be bounded and  $Q(t) = Q(t)^T \geq 0$  for all  $t \geq 0$ . Linear system  $\dot{x} = -Q(t)x$  satisfies  $\lim_{t \rightarrow \infty} x(t) = 0$  if  $Q$  is sufficiently exciting.

Uniform observability, on the other hand, is quite a standard concept, which is more or less equivalent to that the integrand (summand) of the observability grammian is persistently exciting.

To obtain the above listed results we first study synchronization stability of a particular type of array. This array is pertinent to consensus problem (for trajectories are static once synchronized) yet different from the usual array of interest in consensus problems [3]. Our second contribution in this paper is in analyzing this new type of consensus array and, consequently, unraveling two arrays' similarities and differences. In addition, we investigate the stability of their union. The array dynamics generally studied in consensus problems is

$$\dot{x}_i = \sum_{j=1}^p \gamma_{ij}(t)(x_j - x_i) \quad (1)$$

where  $x_i \in \mathbb{R}^n$  is the state of the  $i$ th system and  $\gamma_{ij}(t) \geq 0$  for all  $t$ . What is known about this array is that its trajectories are bounded. In fact, the convex hull of the states  $\text{co}\{x_1, \dots, x_p\}$  is forward invariant regardless of the evolution of  $\gamma_{ij}(\cdot)$ . Moreover, if certain connectedness property is satisfied by the graph described by  $\{\gamma_{ij}\}$ , then trajectories  $x_i(\cdot)$  meet at some common point, i.e. reach consensus. Finally, in general, there does not exist a quadratic Lyapunov function to establish stability; so the convex hull of the states is used instead [9]. The array considered in this paper is

$$\dot{x}_i = \sum_{j=1}^p \gamma_{ij}(y_j - y_i), \quad y_i = Q(t)x_i \quad (2)$$

where time-varying output matrix  $Q(\cdot)$  is symmetric positive semi-definite and  $\gamma_{ij}$  is fixed. Below we list our findings residing in Section 4.

- Like array (1), trajectories of array (2) are bounded.
- Unlike array (1), there exists a quadratic Lyapunov function<sup>4</sup> for array (2).

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<sup>2</sup>Along with these results, we also provide a synchronizing feedback law  $L(\cdot)$  in the paper.

<sup>3</sup>See Definition 2.

<sup>4</sup>However, the convex hull is no longer forward invariant.

- For  $Q$  sufficiently exciting, trajectories of array (2) reach consensus for all connected interconnections. The point of consensus is independent of the evolution of  $Q$ .

We also look at the union of the two cases  $\dot{x}_i = \sum_{j=1}^p \gamma_{ij}(t)(y_j - y_i)$ ,  $y_i = Q(t)x_i$ . We find that unbounded trajectories may result from this situation, hence stability is no longer guaranteed.

The outline of the paper is as follows. After introducing notation and basic definitions, we define synchronizability and give the formal problem statement for continuous-time linear time-varying systems in Section 3. This section also is where we draw the simple link between synchronization of time-varying linear systems and consensus of array (2). In Section 4 we establish the stability of array (2) via a quadratic Lyapunov function and construct (observability) conditions on  $Q$  yielding consensus. In Section 5 we interpret these conditions through the observability grammian of time-varying pair  $(C, A)$  and establish our main results. Finally, in Section 6, we generate the discrete-time versions of the continuous-time results.

## 2 Notation and definitions

Let  $\mathbb{N}$  denote the set of nonnegative integers and  $\mathbb{R}_{\geq 0}$  the set of nonnegative real numbers. The meaning of  $\mathbb{N}_{\geq k}$  is the obvious. Let  $|\cdot|$  denote (induced) 2-norm. Identity matrix in  $\mathbb{R}^{n \times n}$  is denoted by  $I_n$ . The set of all symmetric positive semi-definite (SPSD) matrices in  $\mathbb{R}^{n \times n}$  is denoted by  $\mathcal{Q}_n$ . We also define  $\overline{\mathcal{Q}}_n := \{R \in \mathcal{Q}_n : |R| \leq 1\}$ . Let  $\mathbf{1} \in \mathbb{R}^p$  denote the vector with all entries equal to one. The smallest and largest singular values of  $A \in \mathbb{R}^{m \times n}$  are, respectively, denoted by  $\sigma_{\min}(A)$  and  $\sigma_{\max}(A)$ . *Kronecker product* of  $A \in \mathbb{R}^{m \times n}$  and  $B \in \mathbb{R}^{p \times q}$  is

$$A \otimes B := \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}$$

Kronecker product comes with the following properties:  $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$  (provided that products  $AC$  and  $BD$  are allowed);  $A \otimes B + A \otimes C = A \otimes (B + C)$  (for  $B$  and  $C$  that are of the same size); and  $(A \otimes B)^T = A^T \otimes B^T$ . Moreover, the singular values of  $(A \otimes B)$  equal the (pairwise) product of singular values of  $A$  and  $B$ .

A (*directed*) *graph* is a pair  $(\mathcal{N}, \mathcal{E})$  where  $\mathcal{N}$  is a nonempty finite set (of *nodes*) and  $\mathcal{E}$  is a finite collection of ordered pairs (*edges*)  $(n_i, n_j)$  with  $n_i, n_j \in \mathcal{N}$ . A *directed path* from  $n_1$  to  $n_\ell$  is a sequence of nodes  $(n_1, n_2, \dots, n_\ell)$  such that  $(n_i, n_{i+1})$  is an edge for  $i \in \{1, 2, \dots, \ell - 1\}$ . A graph is *connected* if it has a node to which there exists a directed path from every other node.<sup>5</sup> The graph

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<sup>5</sup>Note that this definition of connectedness for directed graphs is weaker than strong connectivity and stronger than weak connectivity.

of a matrix  $M := [m_{ij}] \in \mathbb{R}^{p \times p}$  is the pair  $(\mathcal{N}, \mathcal{E})$ , where  $\mathcal{N} = \{n_1, n_2, \dots, n_p\}$  and  $\mathcal{E}$  is such that  $(n_i, n_j) \in \mathcal{E}$  iff  $m_{ij} > 0$ . Matrix  $M$  is said to be *connected* when its graph is connected.

Throughout the paper  $\Gamma := [\gamma_{ij}] \in \mathbb{R}^{p \times p}$  will represent an *interconnection* (in the continuous-time sense) satisfying  $\gamma_{ij} \geq 0$  for  $i \neq j$  and  $\gamma_{ii} = -\sum_{j \neq i} \gamma_{ij}$  for all  $i$ . It immediately follows that  $\lambda = 0$  is an eigenvalue with eigenvector  $\mathbf{1}$ , that is,  $\Gamma \mathbf{1} = 0$ . For  $\Gamma$  connected, eigenvalue  $\lambda = 0$  is distinct and all the other eigenvalues have real parts strictly negative. Let  $r \in \mathbb{R}^p$  satisfy

$$r^T \Gamma = 0 \quad (3a)$$

$$r^T \mathbf{1} = 1. \quad (3b)$$

Then  $r$  is unique (for  $\Gamma$  connected) and satisfies  $\lim_{t \rightarrow \infty} e^{\Gamma t} = \mathbf{1} r^T$ . Also,  $r$  has no negative entry. We denote by  $\mathcal{G}_{>0}$  the set of all connected interconnections, i.e.  $\mathcal{G}_{>0} = \{\Gamma \in \mathbb{R}^{p \times p} : \Gamma \text{ connected interconnection, } p = 2, 3, \dots\}$ .

Matrix  $\Lambda := [\lambda_{ij}] \in \mathbb{R}^{p \times p}$  denotes an *interconnection* (in the discrete-time sense) satisfying  $\lambda_{ij} \geq 0$  for all  $i, j$  and  $\sum_j \lambda_{ij} = 1$  for all  $i$ . It follows that  $\lambda = 1$  is an eigenvalue with eigenvector  $\mathbf{1}$ , that is,  $\Lambda \mathbf{1} = \mathbf{1}$ . For a connected  $\Lambda$ , eigenvalue  $\lambda = 1$  is distinct and all the other eigenvalues lie strictly within the unit circle. Let  $r \in \mathbb{R}^p$  satisfy

$$r^T \Lambda = r^T \quad (4a)$$

$$r^T \mathbf{1} = 1. \quad (4b)$$

Then  $r$  is unique (for  $\Lambda$  connected) and satisfies  $\lim_{k \rightarrow \infty} \Lambda^k = \mathbf{1} r^T$ . Also,  $r$  has no negative entry. By slight abuse of notation (yet with a negligible risk of ambiguity) we will let  $\mathcal{G}_{>0}$  also denote the set of all (discrete-time) connected interconnections  $\Lambda$ .

Let  $\mathbb{S} \in \{\mathbb{R}_{\geq 0}, \mathbb{N}\}$ . Given maps  $\xi_i : \mathbb{S} \rightarrow \mathbb{R}^n$  for  $i = 1, 2, \dots, p$  and a map  $\bar{\xi} : \mathbb{S} \rightarrow \mathbb{R}^n$ , the elements of the set  $\{\xi_i(\cdot) : i = 1, 2, \dots, p\}$  are said to *synchronize to*  $\bar{\xi}(\cdot)$  if  $|\xi_i(s) - \bar{\xi}(s)| \rightarrow 0$  as  $s \rightarrow \infty$  for all  $i$ . They are said to *synchronize* if they synchronize to some  $\bar{\xi}(\cdot)$ . Moreover, if there exists a pair of positive real numbers  $(c, \alpha)$  such that  $\max_i |\xi_i(s) - \bar{\xi}(s)| \leq c e^{-\alpha s}$  for all  $s$ , then  $\xi_i(\cdot)$  are said to *exponentially synchronize*.

### 3 Problem statement

For a given interconnection  $\Gamma = [\gamma_{ij}] \in \mathbb{R}^{p \times p}$ , let an array of  $p$  linear systems be

$$\dot{x}_i = A(t)x_i + u_i \quad (5a)$$

$$y_i = C(t)x_i \quad (5b)$$

$$z_i = \sum_{j \neq i} \gamma_{ij}(y_j - y_i) \quad (5c)$$

where  $x_i \in \mathbb{R}^n$  is the *state*,  $u_i \in \mathbb{R}^n$  is the *input*,  $y_i \in \mathbb{R}^m$  is the *output*, and  $z_i \in \mathbb{R}^m$  is the *coupling* of the  $i$ th system for  $i = 1, 2, \dots, p$ . For each  $t \in \mathbb{R}$  we

have  $A(t) \in \mathbb{R}^{n \times n}$  and  $C(t) \in \mathbb{R}^{m \times n}$ . The solution of  $i$ th system at time  $t \geq 0$  is denoted by  $x_i(t)$ . We denote by  $\Phi_A(\cdot, \cdot)$  the state transition matrix for  $A$ , i.e. the unique solution of the matrix differential equation

$$\dot{\Phi}_A(t, t_0) = A(t)\Phi_A(t, t_0)$$

with  $\Phi_A(t_0, t_0) = I_n$ . Also, recall that the observability grammian of pair  $(C, A)$  is given by

$$W_o(t_0, t) := \int_{t_0}^t \Phi_A^T(\tau, t_0) C^T(\tau) C(\tau) \Phi_A(\tau, t_0) d\tau \quad (6)$$

for  $t_0, t \in \mathbb{R}$ . We will henceforth assume that the integrand of the grammian is Riemann-integrable.

**Definition 1 (Synchronizability)** *Given functions  $A : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$  and  $C : \mathbb{R} \rightarrow \mathbb{R}^{m \times n}$ ; pair  $(C, A)$  is said to be synchronizable (with respect to  $\mathcal{G}_{>0}$ ) if there exists a bounded, time-varying linear feedback law  $L : \mathbb{R} \rightarrow \mathbb{R}^{n \times m}$  such that for each  $\Gamma \in \mathcal{G}_{>0}$ , solutions  $x_i(\cdot)$  of array (5) with  $u_i = L(t)z_i$  synchronize for all initial conditions.*

Our objective in this paper is to find sufficient conditions on pair  $(C, A)$ , in particular on the observability grammian (6), for synchronizability and to design a synchronizing feedback law  $L$  when proposed conditions are met.

The above statement of our objective almost suggests that we first find sufficient conditions and search for an  $L$  only *afterwards*. However we adopt the opposite approach. We choose first to construct an  $L$  and then work out the conditions on  $(C, A)$  for synchronization under such feedback law. Given  $(C, A)$  let

$$L(t) := \Phi_A(t, 0) \Phi_A^T(t, 0) C^T(t). \quad (7)$$

For interconnection  $\Gamma \in \mathbb{R}^{p \times p}$  consider array (5) with  $u_i = L(t)z_i$ . We can write

$$\dot{x}_i = A(t)x_i + L(t)C(t) \sum_{j \neq i} \gamma_{ij}(x_j - x_i). \quad (8)$$

Let us define the auxiliary variable  $\xi_i \in \mathbb{R}^n$  as

$$\xi_i(t) := \Phi_A(0, t)x_i(t) \quad (9)$$

for  $i = 1, 2, \dots, p$  and  $t \geq 0$ . Combining (7), (8), and (9) we obtain

$$\dot{\xi}_i = \Phi_A^T(t, 0) C^T(t) C(t) \Phi_A(t, 0) \sum_{j \neq i} \gamma_{ij}(\xi_j - \xi_i). \quad (10)$$

Now note that if  $\Phi_A$  is bounded, then synchronization of solutions  $\xi_i(\cdot)$  implies synchronization of solutions  $x_i(\cdot)$  by (9). Moreover, if  $C$  is bounded as well, then boundedness of  $L$  is guaranteed by (7). Based on this simple observation let us write the following assumption to be invoked later.

**Assumption 1 (Boundedness)** For  $A : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$  and  $C : \mathbb{R} \rightarrow \mathbb{R}^{m \times n}$  following hold.

- (a) There exists  $\bar{a} \geq 1$  such that  $|\Phi_A(t_1, t_2)| \leq \bar{a}$  for all  $t_1, t_2 \geq 0$ .
- (b) There exists  $\bar{c} \geq 1$  such that  $|C(t)| \leq \bar{c}$  for all  $t \geq 0$ .

**Remark 1** Note that in the time-invariant case Assumption 1(b) comes for free and Assumption 1(a) boils down to that matrix  $A$  is neutrally stable (in the continuous-time sense) with all its eigenvalues residing on the imaginary axis.

The second point we want to make is that the term multiplying the sum in (10) is the integrand of the observability grammian, which is SPSD at each  $t$ . We elaborate on this fact in the next section.

## 4 Synchronization under SPSD matrix

For a given interconnection  $\Gamma = [\gamma_{ij}] \in \mathbb{R}^{p \times p}$ , let an array of  $p$  systems be

$$\dot{x}_i = Q_t \sum_{j \neq i} \gamma_{ij} (x_j - x_i) \quad (11)$$

where  $x_i \in \mathbb{R}^n$  is the state of the  $i$ th system (for  $i = 1, 2, \dots, p$ ) and  $Q_t \in \mathbb{R}^{n \times n}$  is SPSD for each  $t \geq 0$ . We assume  $Q : \mathbb{R}_{\geq 0} \rightarrow \mathcal{Q}_n$  to be Riemann-integrable. By letting

$$\mathbf{x} := \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix}$$

we can rewrite (11) more compactly as

$$\dot{\mathbf{x}} = (\Gamma \otimes Q_t) \mathbf{x}. \quad (12)$$

**Remark 2** Sometimes we need function  $Q : \mathbb{R}_{\geq 0} \rightarrow \mathcal{Q}_n$  be bounded on the interval  $[0, \infty)$ , i.e. there exists  $h \geq 1$  such that  $|Q_t| \leq h$  for all  $t$ . Note that (12) can be written as

$$\dot{\mathbf{x}} = \left( h\Gamma \otimes \frac{Q_t}{h} \right) \mathbf{x}.$$

Now, since  $\Gamma$  is an interconnection, so is  $h\Gamma$ . Also, connectedness is invariant under multiplication by a positive scalar, i.e.  $\Gamma$  is connected if and only if  $h\Gamma$  is. Finally, observe that  $Q_t/h \in \overline{\mathcal{Q}}_n$ . Without loss of generality (for our purposes) therefore we can take  $h$  to be unity, which lets us consider  $Q : \mathbb{R}_{\geq 0} \rightarrow \overline{\mathcal{Q}}_n$  whenever we need  $Q$  be bounded.

In the rest of this section we first show that the origin of system (12) is stable regardless of interconnection  $\Gamma \in \mathbb{R}^{p \times p}$  and function  $Q : \mathbb{R} \rightarrow \mathcal{Q}_n$ . Then, under connectedness of  $\Gamma$ , which is obviously necessary for synchronization, we work out some sufficient conditions on function  $Q$  to establish synchronization of solutions  $x_i(\cdot)$  of array (11). Finally, we provide two theorems to make the picture that we want to give in this section closer to complete. One of those theorems states that time-invariance of interconnection  $\Gamma$  in (12) is necessary for stability. With the other one, we aim to show that the sufficient conditions that we will have proposed on  $Q$  for synchronization cannot be *readily* relaxed into a less technical one without sacrificing generality.

#### 4.1 Stability

**Lemma 1** *Let interconnection  $\Gamma \in \mathbb{R}^{p \times p}$  be connected and  $r \in \mathbb{R}^p$  satisfy (3). Then, there exists symmetric positive definite matrix  $\Omega \in \mathbb{R}^{p \times p}$  such that*

$$(\Gamma - \mathbf{1}r^T)^T \Omega + \Omega(\Gamma - \mathbf{1}r^T) = -I_p. \quad (13)$$

**Proof.** Consider matrix  $\Gamma - \mathbf{1}r^T$ . Observe that  $(\Gamma - \mathbf{1}r^T)^k = \Gamma^k + (-1)^k \mathbf{1}r^T$  for  $k \in \mathbb{N}$ . For  $t \in \mathbb{R}$ , therefore we can write

$$\begin{aligned} e^{(\Gamma - \mathbf{1}r^T)t} &= I_p + t(\Gamma - \mathbf{1}r^T) + \frac{t^2}{2}(\Gamma - \mathbf{1}r^T)^2 + \dots \\ &= \left( I_p + t\Gamma + \frac{t^2}{2}\Gamma^2 + \dots \right) - \left( t\mathbf{1}r^T - \frac{t^2}{2}\mathbf{1}r^T + \dots \right) \\ &= e^{\Gamma t} - (1 - e^{-t})\mathbf{1}r^T. \end{aligned}$$

Consequently,  $\lim_{t \rightarrow \infty} e^{(\Gamma - \mathbf{1}r^T)t} = 0$ ; and we deduce that  $[\Gamma - \mathbf{1}r^T]$  is Hurwitz. Therefore Lyapunov equation (13) admits a symmetric positive definite solution  $\Omega$ .  $\blacksquare$

**Lemma 2** *Let interconnection  $\Gamma \in \mathbb{R}^{p \times p}$  be connected,  $r \in \mathbb{R}^p$  satisfy (3), and symmetric positive definite matrix  $\Omega \in \mathbb{R}^{p \times p}$  satisfy (13). Define  $V : \mathbb{R}^{np} \rightarrow \mathbb{R}_{\geq 0}$  as  $V(\mathbf{x}) := \mathbf{x}^T(\Omega \otimes I_n)\mathbf{x}$ . Then, for all  $Q : \mathbb{R}_{\geq 0} \rightarrow \mathcal{Q}_n$  and all  $t \geq 0$ , solution of system (12) satisfies*

$$\frac{d}{dt}V(\mathbf{x}(t) - \bar{\mathbf{x}}) = -(\mathbf{x}(t) - \bar{\mathbf{x}})^T(I_p \otimes Q_t)(\mathbf{x}(t) - \bar{\mathbf{x}})$$

where  $\bar{\mathbf{x}} := (\mathbf{1}r^T \otimes I_n)\mathbf{x}(0)$ .

**Proof.** Observe that  $(\mathbf{1}r^T \otimes I_n)\dot{\mathbf{x}}(t) = 0$ , which implies  $(\mathbf{1}r^T \otimes I_n)\mathbf{x}(t) = \bar{\mathbf{x}}$  for all  $t \geq 0$ . Whence  $\dot{\mathbf{x}}(t) = ((\Gamma - \mathbf{1}r^T) \otimes Q_t)(\mathbf{x}(t) - \bar{\mathbf{x}})$ . We can therefore write

$$\begin{aligned} \frac{d}{dt}V(\mathbf{x}(t) - \bar{\mathbf{x}}) &= \dot{\mathbf{x}}(t)^T(\Omega \otimes I_n)(\mathbf{x}(t) - \bar{\mathbf{x}}) + (\mathbf{x}(t) - \bar{\mathbf{x}})^T(\Omega \otimes I_n)\dot{\mathbf{x}}(t) \\ &= (\mathbf{x}(t) - \bar{\mathbf{x}})^T((\Gamma - \mathbf{1}r^T) \otimes Q_t)^T(\Omega \otimes I_n)(\mathbf{x}(t) - \bar{\mathbf{x}}) \\ &\quad + (\mathbf{x}(t) - \bar{\mathbf{x}})^T(\Omega \otimes I_n)((\Gamma - \mathbf{1}r^T) \otimes Q_t)(\mathbf{x}(t) - \bar{\mathbf{x}}) \\ &= (\mathbf{x}(t) - \bar{\mathbf{x}})^T [((\Gamma - \mathbf{1}r^T)^T \Omega + \Omega(\Gamma - \mathbf{1}r^T)) \otimes Q_t] (\mathbf{x}(t) - \bar{\mathbf{x}}) \\ &= -(\mathbf{x}(t) - \bar{\mathbf{x}})^T(I_p \otimes Q_t)(\mathbf{x}(t) - \bar{\mathbf{x}}). \end{aligned}$$

Hence the result. ■

**Theorem 1 (Stability)** *Given interconnection  $\Gamma \in \mathbb{R}^{p \times p}$ , there exists  $\alpha > 0$  such that, for all  $Q : \mathbb{R}_{\geq 0} \rightarrow \mathcal{Q}_n$ , solution of system (12) satisfies*

$$|\mathbf{x}(t)| \leq \alpha |\mathbf{x}(0)|$$

for all  $t \geq 0$ .

**Proof.** Interconnection  $\Gamma$  is similar to a block diagonal matrix  $\text{diag}(\Gamma_1, \Gamma_2, \dots, \Gamma_q)$  in  $\mathbb{R}^{p \times p}$  such that  $\Gamma_i \in \mathbb{R}^{p_i \times p_i}$  for  $i = 1, 2, \dots, q$  is a connected interconnection if  $p_i \geq 2$  and  $\Gamma_i = 0$  otherwise. (Integer  $q$  equals the number of eigenvalues of  $\Gamma$  at the origin.) Since  $\text{diag}(\Gamma_1, \Gamma_2, \dots, \Gamma_q) \otimes Q_t = \text{diag}(\Gamma_1 \otimes Q_t, \Gamma_2 \otimes Q_t, \dots, \Gamma_q \otimes Q_t)$  without loss of generality it suffices to check two cases: (i)  $\Gamma = 0$ ; and (ii)  $\Gamma$  is connected. First case is trivial; so let us suppose  $\Gamma$  is connected.

Now, let  $r \in \mathbb{R}^p$  and symmetric positive definite matrix  $\Omega \in \mathbb{R}^{p \times p}$  satisfy (3) and (13), respectively. Given  $Q : \mathbb{R}_{\geq 0} \rightarrow \mathcal{Q}_n$ , consider system (12). Let  $\bar{\mathbf{x}} = (\mathbf{1} r^T \otimes I_n) \mathbf{x}(0)$ . Recalling that  $r$  has no negative entry, we can write

$$\begin{aligned} |\bar{\mathbf{x}}| &\leq |\mathbf{1} r^T| |\mathbf{x}(0)| \\ &\leq |\mathbf{1}| |\mathbf{x}(0)| \\ &= \sqrt{n} |\mathbf{x}(0)|. \end{aligned}$$

Let  $V(\mathbf{x}) = \mathbf{x}^T (\Omega \otimes I_n) \mathbf{x}$ . Lemma 2 yields that  $V(\mathbf{x}(\cdot) - \bar{\mathbf{x}})$  is nonincreasing. Hence

$$\begin{aligned} |\mathbf{x}(t)| &\leq |\mathbf{x}(t) - \bar{\mathbf{x}}| + |\bar{\mathbf{x}}| \\ &\leq \frac{1}{\sqrt{\sigma_{\min}(\Omega)}} \sqrt{V(\mathbf{x}(t) - \bar{\mathbf{x}})} + |\bar{\mathbf{x}}| \\ &\leq \frac{1}{\sqrt{\sigma_{\min}(\Omega)}} \sqrt{V(\mathbf{x}(0) - \bar{\mathbf{x}})} + |\bar{\mathbf{x}}| \\ &\leq \sqrt{\frac{\sigma_{\max}(\Omega)}{\sigma_{\min}(\Omega)}} |\mathbf{x}(0) - \bar{\mathbf{x}}| + |\bar{\mathbf{x}}| \\ &\leq \left( \sqrt{\frac{\sigma_{\max}(\Omega)}{\sigma_{\min}(\Omega)}} (1 + \sqrt{n}) + \sqrt{n} \right) |\mathbf{x}(0)| \end{aligned}$$

for all  $t \geq 0$ . ■

Theorem 1 establishes stability. That is, for a fixed interconnection<sup>6</sup>  $\Gamma$ , which need not be connected, solutions  $x_i(\cdot)$  of array (11) stay within a bounded region (that depends only on initial conditions  $x_i(0)$  and interconnection  $\Gamma$ ) for all time-varying SPSD matrix  $Q$ . Whenever  $\Gamma$  is connected, that bounded region can be described quite precisely. See the next result, which is a direct consequence of Lemma 2.

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<sup>6</sup>Later in the section, we will also investigate whether stability is preserved when both  $\Gamma$  and  $Q$  are time-varying.

**Theorem 2** *Given connected interconnection  $\Gamma \in \mathbb{R}^{p \times p}$ , let  $r \in \mathbb{R}^p$  and symmetric positive definite matrix  $\Omega \in \mathbb{R}^{p \times p}$  satisfy (3) and (13), respectively. Then, for all  $Q : \mathbb{R}_{\geq 0} \rightarrow \mathcal{Q}_n$ , solutions  $x_i(\cdot)$  of array (11) satisfy, for all  $t \geq 0$ ,*

$$|x_i(t) - \bar{x}| \leq \left( \frac{\sigma_{\max}(\Omega)}{\sigma_{\min}(\Omega)} \sum_{j=1}^p |x_j(0) - \bar{x}|^2 \right)^{1/2}$$

where  $\bar{x} := (r^T \otimes I_n) \mathbf{x}(0)$ .

## 4.2 Asymptotic synchronization

We now begin looking for sufficient conditions on  $Q : \mathbb{R} \rightarrow \overline{\mathcal{Q}}_n$  that guarantee that solutions  $x_i(\cdot)$  of array (11) synchronize. The next fact is to be used by the key theorem following it.

**Fact 1** *Let  $f : [0, T] \rightarrow [0, 1]$  be Riemann-integrable. Then*

$$3T^2 \int_0^T f^2(t) dt \geq \left( \int_0^T f(t) dt \right)^3.$$

**Proof.** Result trivially follows for  $T = 0$ . Suppose  $T > 0$ . Fix some  $\delta > 0$  such that  $T/\delta =: N$  is an integer and let  $I_f := \int_{[0, T]} f$ . Then there exists  $k_1 \in \{1, 2, \dots, N\}$  and  $t_1 \in [(k_1 - 1)\delta, k_1\delta]$  such that  $f(t_1) \geq I_f/T$ . Since  $f(t) \leq 1$  for all  $t$ , we can also claim that there exists  $k_2 \in \{1, 2, \dots, N\} \setminus \{k_1\}$  and  $t_2 \in [(k_2 - 1)\delta, k_2\delta]$  such that  $f(t_2) \geq (I_f - \delta)/T$ . Following the pattern, we can generate a sequence  $(k_i)_{i=1}^{\lfloor I_f/\delta \rfloor}$  of distinct elements from the set  $\{1, 2, \dots, N\}$  and an associated sequence  $(t_i)_{i=1}^{\lfloor I_f/\delta \rfloor}$  satisfying  $t_i \in [(k_i - 1)\delta, k_i\delta]$  and  $f(t_i) \geq (I_f - (i - 1)\delta)/T$ . Therefore we can write

$$\begin{aligned} \sum_{k=1}^{\lfloor T/\delta \rfloor} \sup_{t \in [(k-1)\delta, k\delta]} f^2(t) \delta &\geq \sum_{i=1}^{\lfloor I_f/\delta \rfloor} f^2(t_i) \delta \\ &\geq \delta \sum_{i=1}^{\lfloor I_f/\delta \rfloor} \left( \frac{I_f - (i-1)\delta}{T} \right)^2. \end{aligned}$$

By definition of integral, we can therefore write

$$\begin{aligned}
\int_0^T f^2(t)dt &= \lim_{\delta \rightarrow 0^+} \sum_{k=1}^{\lfloor T/\delta \rfloor} \sup_{t \in [(k-1)\delta, k\delta]} f^2(t) \delta \\
&\geq \lim_{\delta \rightarrow 0^+} \delta \sum_{i=1}^{\lfloor I_f/\delta \rfloor} \left( \frac{I_f - (i-1)\delta}{T} \right)^2 \\
&= \lim_{\delta \rightarrow 0^+} \frac{\delta}{T^2} \sum_{i=1}^{\lfloor I_f/\delta \rfloor} (I_f - i\delta)^2 \\
&= \lim_{\delta \rightarrow 0^+} \frac{\delta}{T^2} \left( \sum_{i=1}^{\lfloor I_f/\delta \rfloor} (I_f^2 - 2I_f i\delta) + \sum_{i=1}^{\lfloor I_f/\delta \rfloor} i^2 \delta^2 \right) \\
&= \lim_{\delta \rightarrow 0^+} \frac{\delta^3}{T^2} \sum_{i=1}^{\lfloor I_f/\delta \rfloor} i^2 \\
&= \frac{I_f^3}{3T^2}.
\end{aligned}$$

Hence the result. ■

**Theorem 3** *Given a pair of positive real numbers  $(\varepsilon, T)$ , define*

$$\delta(\varepsilon, T) := \min \left\{ \frac{\varepsilon}{2}, \frac{\varepsilon^3}{240T^5} \right\}. \quad (14)$$

*Given connected interconnection  $\Gamma \in \mathbb{R}^{p \times p}$ , let  $r \in \mathbb{R}^p$  and symmetric positive definite matrix  $\Omega \in \mathbb{R}^{p \times p}$  satisfy (3) and (13), respectively. Define*

$$\rho(\Gamma) := \sigma_{\max}(\Omega) \max\{1, |\Gamma|^3\}. \quad (15)$$

*Let  $V(\mathbf{x}) := \mathbf{x}^T (\Omega \otimes I_n) \mathbf{x}$  for  $\mathbf{x} \in \mathbb{R}^{np}$ . Then, for all  $Q : \mathbb{R}_{\geq 0} \rightarrow \overline{\mathcal{Q}}_n$ , the below inequality*

$$\sigma_{\min} \left( \int_0^T Q_t dt \right) \geq \varepsilon \quad (16)$$

*implies that solution  $\mathbf{x}(\cdot)$  of system (12) satisfies*

$$V(\mathbf{x}(T) - \bar{\mathbf{x}}) \leq \left( 1 - \frac{\delta(\varepsilon, T)}{\rho(\Gamma)} \right) V(\mathbf{x}(0) - \bar{\mathbf{x}})$$

*where  $\bar{\mathbf{x}} := (\mathbf{1}r^T \otimes I_n) \mathbf{x}(0)$ .*

**Proof.** Given pair  $(\varepsilon, T)$  let  $\omega := \varepsilon/(4T)$ . Consider system (12). Let us introduce

$$\xi(t) := \mathbf{x}(t) - \bar{\mathbf{x}}.$$

By Lemma 2, we have

$$\dot{V}(\xi(t)) = -\xi^T(t)(I_p \otimes Q_t)\xi(t). \quad (17)$$

Also,  $\xi(\cdot)$  can be shown to satisfy

$$\dot{\xi} = (\Gamma \otimes Q_t)\xi. \quad (18)$$

Let  $Q : \mathbb{R}_{\geq 0} \rightarrow \overline{Q}_n$  satisfy (16). Then, regarding the evolution of  $\xi(\cdot)$ , one of the two following cases must be.

*Case 1:*  $|\xi(t) - \xi(0)| \leq \omega|\xi(0)|$  for all  $t \in [0, T]$ . Let  $b(t) := \xi(t) - \xi(0)$  and recall that  $|Q_t| \leq 1$ . For reasons of economy, let us adopt the notation  $\mathbf{Q}_t := (I_p \otimes Q_t)$ . Note that then we have  $|\mathbf{Q}_t| \leq 1$  as well as

$$\sigma_{\min} \left( \int_0^T \mathbf{Q}_t dt \right) \geq \varepsilon.$$

(Observe that  $T \geq \varepsilon$ .) From (17) we can write

$$\begin{aligned} V(\xi(T)) &= V(\xi(0)) - \int_0^T \xi^T(t) \mathbf{Q}_t \xi(t) dt \\ &= V(\xi(0)) - \int_0^T (\xi(0) + b(t))^T \mathbf{Q}_t (\xi(0) + b(t)) dt \\ &\leq V(\xi(0)) - \xi^T(0) \left[ \int_0^T \mathbf{Q}_t dt \right] \xi(0) - 2 \int_0^T b^T(t) \mathbf{Q}_t \xi(0) dt - \int_0^T b^T(t) \mathbf{Q}_t b(t) dt \\ &\leq V(\xi(0)) - \varepsilon |\xi(0)|^2 + 2\omega T |\xi(0)|^2 \\ &= V(\xi(0)) - \frac{\varepsilon}{2} |\xi(0)|^2 \\ &\leq \left( 1 - \frac{\varepsilon}{2\sigma_{\max}(\Omega)} \right) V(\xi(0)). \end{aligned} \quad (19)$$

*Case 2:*  $|\xi(\bar{t}) - \xi(0)| = \omega|\xi(0)|$  for some  $\bar{t} \in (0, T]$ . Without loss of generality, assume  $|\xi(t) - \xi(0)| < \omega|\xi(0)|$  for  $t \in [0, \bar{t})$ . We can by (18) write

$$\begin{aligned} \int_0^{\bar{t}} |\mathbf{Q}_t \xi(t)| dt &= |\Gamma|^{-1} \int_0^{\bar{t}} |\Gamma \otimes I_n| |\mathbf{Q}_t \xi(t)| dt \\ &\geq |\Gamma|^{-1} \int_0^{\bar{t}} |(\Gamma \otimes Q_t) \xi(t)| dt \\ &\geq |\Gamma|^{-1} \left| \int_0^{\bar{t}} (\Gamma \otimes Q_t) \xi(t) dt \right| \\ &= |\Gamma|^{-1} \omega |\xi(0)|. \end{aligned} \quad (20)$$

Since  $|\mathbf{Q}_t \xi(t)| \leq (1 + \omega)|\xi(0)|$  for  $t \in [0, \bar{t}]$ , we can invoke Fact 1 on (20) and obtain

$$\begin{aligned}
\int_0^{\bar{t}} |\mathbf{Q}_t \xi(t)|^2 dt &= (1 + \omega)^2 |\xi(0)|^2 \int_0^{\bar{t}} \left( \frac{|\mathbf{Q}_t \xi(t)|}{(1 + \omega)|\xi(0)|} \right)^2 dt \\
&\geq (1 + \omega)^2 |\xi(0)|^2 \left( \frac{\omega}{(1 + \omega)|\Gamma|} \right)^3 \frac{1}{3\bar{t}^2} \\
&\geq (1 + \omega)^2 |\xi(0)|^2 \left( \frac{\omega}{(1 + \omega)|\Gamma|} \right)^3 \frac{1}{3T^2} \\
&= \frac{\omega^3}{3(1 + \omega)T^2|\Gamma|^3} |\xi(0)|^2.
\end{aligned} \tag{21}$$

Since  $|\mathbf{Q}_t| \leq 1$ , we have  $\mathbf{Q}_t^{1/2} \geq \mathbf{Q}_t$ . Also, note that  $V(\xi(t))$  is nonincreasing thanks to (17). Now, by (21) we can write

$$\begin{aligned}
V(\xi(T)) &= V(\xi(0)) - \int_0^T \xi^T(t) \mathbf{Q}_t \xi(t) dt \\
&\leq V(\xi(0)) - \int_0^{\bar{t}} \xi^T(t) \mathbf{Q}_t \xi(t) dt \\
&= V(\xi(0)) - \int_0^{\bar{t}} |\mathbf{Q}_t^{1/2} \xi(t)|^2 dt \\
&\leq V(\xi(0)) - \int_0^{\bar{t}} |\mathbf{Q}_t \xi(t)|^2 dt \\
&\leq V(\xi(0)) - \frac{\omega^3}{3(1 + \omega)T^2|\Gamma|^3} |\xi(0)|^2 \\
&\leq \left( 1 - \frac{\varepsilon^3}{48T^4(4T + \varepsilon)|\Gamma|^3 \sigma_{\max}(\Omega)} \right) V(\xi(0)) \\
&\leq \left( 1 - \frac{\varepsilon^3}{240T^5|\Gamma|^3 \sigma_{\max}(\Omega)} \right) V(\xi(0)).
\end{aligned} \tag{22}$$

The result follows by (14), (15), (19), and (22).  $\blacksquare$

**Fact 2** Let  $(a_i)_{i=1}^{\infty}$  be a sequence with  $0 \leq a_i < 1$  for all  $i$ . Then, product  $\prod_{i=1}^{\infty} (1 - a_i)$  converges to zero if and only if sum  $\sum_{i=1}^{\infty} a_i$  diverges.

**Proof.** See e.g. [12, Thm. 1.17 of Ch. VII].  $\blacksquare$

In the light of Theorem 3 and Fact 2, we now state our most general condition on  $Q : \mathbb{R} \rightarrow \overline{\mathcal{Q}}_n$  for synchronization of solutions  $x_i(\cdot)$  of array (11).

**Definition 2** Function  $Q : \mathbb{R}_{\geq 0} \rightarrow \mathcal{Q}_n$  is said to be sufficiently exciting if there exists a sequence of pairs of positive real numbers  $(\varepsilon_i, T_i)_{i=1}^{\infty}$  satisfying

$$\sigma_{\min} \left( \int_{t_i}^{t_i + T_i} Q_t dt \right) \geq \varepsilon_i \tag{23}$$

for  $t_i = \sum_{j=1}^{i-1} T_j$  with  $t_1 = 0$ , and

$$\lim_{N \rightarrow \infty} \sum_{i=1}^N \delta(\varepsilon_i, T_i) = \infty \quad (24)$$

where  $\delta(\cdot, \cdot)$  is as defined in (14).

**Theorem 4** Let  $Q : \mathbb{R}_{\geq 0} \rightarrow \overline{\mathcal{Q}}_n$  be sufficiently exciting. Then, for all connected interconnection  $\Gamma \in \mathbb{R}^{p \times p}$ , solutions  $x_i(\cdot)$  of array (11) synchronize to

$$\bar{x}(t) \equiv (r^T \otimes I_n) \mathbf{x}(0)$$

where  $r \in \mathbb{R}^p$  satisfies (3).

**Proof.** Let us be given function  $Q : \mathbb{R}_{\geq 0} \rightarrow \overline{\mathcal{Q}}_n$  and sequence  $(\varepsilon_i, T_i)$  satisfying (23) and (24). Let us let  $\delta_i := \delta(\varepsilon_i, T_i)$ . Let interconnection  $\Gamma \in \mathbb{R}^{p \times p}$  be connected,  $r \in \mathbb{R}^p$  satisfy (3), and symmetric positive definite matrix  $\Omega \in \mathbb{R}^{p \times p}$  satisfy (13). Consider system (12) and let  $\bar{\mathbf{x}} = (\mathbf{1} r^T \otimes I_n) \mathbf{x}(0)$ . Now, letting  $V(\mathbf{x}) = \mathbf{x}^T (\Omega \otimes I_n) \mathbf{x}$ , by Theorem 3 we can write

$$V(\mathbf{x}(t_{i+1}) - \bar{\mathbf{x}}) \leq \left(1 - \frac{\delta_i}{\rho(\Gamma)}\right) V(\mathbf{x}(t_i) - \bar{\mathbf{x}})$$

for  $i = 1, 2, \dots$  where  $\rho(\cdot)$  is as defined in (15). Whence

$$V(\mathbf{x}(t_i) - \bar{\mathbf{x}}) \leq V(\mathbf{x}(0) - \bar{\mathbf{x}}) \prod_{j=1}^{i-1} \left(1 - \frac{\delta_j}{\rho(\Gamma)}\right). \quad (25)$$

Let now  $a_i := \delta_i / \rho(\Gamma)$ . We can write by (24)

$$\begin{aligned} \lim_{N \rightarrow \infty} \sum_{i=1}^N a_i &= \rho(\Gamma)^{-1} \lim_{N \rightarrow \infty} \sum_{i=1}^N \delta_i \\ &= \infty. \end{aligned}$$

Now, we can invoke Fact 2 and claim that  $\lim_{N \rightarrow \infty} \prod_{i=1}^N (1 - a_i) = 0$ , which yields by (25)

$$\lim_{i \rightarrow \infty} V(\mathbf{x}(t_i) - \bar{\mathbf{x}}) = 0.$$

Hence the result. ■

**Corollary 1** Let  $Q : \mathbb{R}_{\geq 0} \rightarrow \overline{\mathcal{Q}}_n$  be sufficiently exciting. Then solution to linear system  $\dot{x} = -Q_t x$  satisfies  $\lim_{t \rightarrow \infty} x(t) = 0$ .

The following definition is quite standard; see, for instance, [8]. Note that the condition it depicts is less general than that of Definition 2, yet it guarantees exponential synchronization.

**Definition 3** Map  $Q : \mathbb{R}_{\geq 0} \rightarrow \mathcal{Q}_n$  is said to be persistently exciting if there exists a pair of positive real numbers  $(\varepsilon, T)$  such that

$$\sigma_{\min} \left( \int_t^{t+T} Q_\tau d\tau \right) \geq \varepsilon \quad (26)$$

for all  $t \geq 0$ .

**Remark 3** The following theorem can be viewed as a generalization of a classic result in adaptive control theory [13, Thm. 2.5.1]. Note that Corollary 1 makes another generalization to this result since “sufficiently exciting” is weaker than “persistently exciting”.

**Theorem 5** Let interconnection  $\Gamma \in \mathbb{R}^{p \times p}$  be connected and function  $Q : \mathbb{R}_{\geq 0} \rightarrow \overline{\mathcal{Q}}_n$  be persistently exciting. Then, solutions  $x_i(\cdot)$  of array (11) exponentially synchronize to

$$\bar{x}(t) \equiv (r^T \otimes I_n) \mathbf{x}(0)$$

where  $r \in \mathbb{R}^p$  satisfies (3).

**Proof.** Since  $Q : \mathbb{R}_{\geq 0} \rightarrow \overline{\mathcal{Q}}_n$  is persistently exciting, by definition, there exists a pair of positive real numbers  $(\varepsilon, T)$  satisfying (26) for all  $t \geq 0$ . Let  $V(\mathbf{x}) = \mathbf{x}^T (\Omega \otimes I_n) \mathbf{x}$  for  $\mathbf{x} \in \mathbb{R}^{np}$ , where symmetric positive definite matrix  $\Omega \in \mathbb{R}^{p \times p}$  satisfy (13). Then, by Theorem 3 we can write

$$V(\mathbf{x}(kT) - \bar{\mathbf{x}}) \leq \left( 1 - \frac{\delta(\varepsilon, T)}{\rho(\Gamma)} \right)^k V(\mathbf{x}(0) - \bar{\mathbf{x}})$$

for all  $k \in \mathbb{N}$ , where  $\bar{\mathbf{x}} := (\mathbf{1} r^T \otimes I_n) \mathbf{x}(0)$ . The result then follows.  $\blacksquare$

We now present an interesting application of Theorem 5 on coupled harmonic oscillators (in  $\mathbb{R}^2$ ) described by

$$\begin{aligned} \dot{x}_{i1} &= x_{i2} \\ \dot{x}_{i2} &= -x_{i1} + \sum_{j \neq i} \gamma_{ij} (x_{j2} - x_{i2}) \end{aligned}$$

for  $i = 1, 2, \dots, p$ . Let

$$x_i = \begin{bmatrix} x_{i1} \\ x_{i2} \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad C = [0 \ 1].$$

Then we can write

$$\dot{x}_i = Ax_i + C^T C \sum_{j \neq i} \gamma_{ij} (x_j - x_i).$$

Define  $\xi_i(t) := e^{-At} x_i(t)$  and  $\xi = [\xi_1^T \ \dots \ \xi_p^T]^T$ . Then

$$\dot{\xi} = (\Gamma \otimes e^{A^T t} C^T C e^{At}) \xi$$

for  $A$  is skew-symmetric. A trivial computation shows

$$e^{A^T t} C^T C e^{A t} = \begin{bmatrix} \cos^2 t & -\sin t \cos t \\ -\sin t \cos t & \sin^2 t \end{bmatrix} \in \overline{\mathcal{Q}}_2$$

whence

$$\int_t^{t+2\pi} e^{A^T \tau} C^T C e^{A \tau} d\tau = \pi I_2$$

for all  $t$ . Therefore  $t \mapsto e^{A^T t} C^T C e^{A t}$  is persistently exciting. Now, suppose that  $\Gamma$  is connected and  $r \in \mathbb{R}^p$  satisfies (3). Then, by Theorem 5 solutions  $\xi_i(\cdot)$  exponentially synchronize to  $\bar{\xi}(t) \equiv (r^T \otimes I_2)\xi(0)$ . Since  $x_i(t) = e^{A t} \xi_i(t)$  and  $e^{A t}$  is an orthogonal (hence norm-preserving) matrix, it follows that solutions  $x_i(\cdot)$  of the coupled harmonic oscillators exponentially synchronize to

$$\bar{x}(t) = (r^T \otimes e^{A t}) \begin{bmatrix} x_1(0) \\ \vdots \\ x_p(0) \end{bmatrix}$$

### 4.3 Negative results

Before we end this section, we present two negative results, which we believe constitute answers to questions that arise naturally. The first of those questions emerges as follows. In Theorem 1 we have proven that system  $\dot{\mathbf{x}} = (\Gamma \otimes Q_t)\mathbf{x}$ , where  $\Gamma$  is a *fixed* interconnection and  $Q$  is a time-varying SPSD matrix, has a bounded solution for all initial conditions. It also trivially follows from the results in, for instance, [9, 7] that solution of system  $\dot{\mathbf{x}} = (\Gamma_t \otimes Q)\mathbf{x}$ , where this time interconnection  $\Gamma$  is time-varying and SPSD matrix  $Q$  is fixed, is bounded. At this point, it is tempting to ask the following question.

*Is solution of system  $\dot{\mathbf{x}} = (\Gamma_t \otimes Q_t)\mathbf{x}$  bounded?*

The answer is *not always* and it is formalized in the below result.

**Theorem 6** *There exist maps  $\Gamma : \mathbb{R} \rightarrow \mathbb{R}^{p \times p}$ , where  $\Gamma_t$  is an interconnection for each  $t$ , and  $Q : \mathbb{R} \rightarrow \overline{\mathcal{Q}}_n$  such that system  $\dot{\mathbf{x}} = (\Gamma_t \otimes Q_t)\mathbf{x}$  has an unbounded solution.*

**Proof.** We construct  $\Gamma$  and  $Q$  as follows. Let

$$\Gamma_a := \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad \Gamma_b := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Gamma_c := \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}, \quad \Gamma_d := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$Q_a := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q_b := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad Q_c := \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}, \quad Q_d := \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}$$

Now let both  $\Gamma$  and  $Q$  be periodic with period  $T = 40$  with

$$\Gamma_t := \begin{cases} \Gamma_a & \text{for } 0 \leq t < 10 \\ \Gamma_b & \text{for } 10 \leq t < 20 \\ \Gamma_c & \text{for } 20 \leq t < 30 \\ \Gamma_d & \text{for } 30 \leq t < 40 \end{cases} \quad \text{and} \quad Q_t := \begin{cases} Q_a & \text{for } 0 \leq t < 10 \\ Q_b & \text{for } 10 \leq t < 20 \\ Q_c & \text{for } 20 \leq t < 30 \\ Q_d & \text{for } 30 \leq t < 40 \end{cases}$$

Whence we can write for  $k = 0, 1, \dots$

$$\mathbf{x}(kT) = \mathbf{A}^k \mathbf{x}(0)$$

where  $\mathbf{A} := e^{(\Gamma_d \otimes Q_d)10} e^{(\Gamma_c \otimes Q_c)10} e^{(\Gamma_b \otimes Q_b)10} e^{(\Gamma_a \otimes Q_a)10}$ . When the eigenvalues of  $\mathbf{A}$  are numerically checked, one finds that there is an eigenvalue outside the unit circle ( $|\lambda| \approx 2$ ), which lets us deduce that the origin of system  $\dot{\mathbf{x}} = (\Gamma_t \otimes Q_t)\mathbf{x}$  is unstable. ■

Our first question was concerned with stability under time-varying interconnection; and we have seen that solutions  $x_i(\cdot)$  of array (11) need not stay bounded in such a case. The second question is about synchronization. By Theorem 4 we know that if map  $Q : \mathbb{R}_{\geq 0} \rightarrow \overline{\mathcal{Q}}_n$  is sufficiently exciting, then for all connected interconnection  $\Gamma$  solutions of array (11) synchronize. Now, suppose that we are given some sufficiently exciting  $Q$  with an associated sequence  $(\varepsilon_i, T_i)_{i=1}^\infty$ , see Definition 2. Note that, due to (14), we have  $\sum_{i=1}^\infty \varepsilon_i = \infty$ . In addition, since  $Q_t \in \overline{\mathcal{Q}}_n$  for all  $t$ , we have  $T_i \geq \varepsilon_i$ , which yields  $\sum_{i=1}^\infty T_i = \infty$ . Applying these observations on (23), we obtain

$$\lim_{T \rightarrow \infty} \sigma_{\min} \left( \int_0^T Q_t dt \right) = \infty. \quad (27)$$

Above condition, depicted in (27), can be shown to be necessary for synchronization. Now, we ask the following question.

*Is condition (27) sufficient for synchronization?*

The answer turns out to be negative. In fact, even a much stronger condition is not sufficient for synchronization as the following result shows.

**Theorem 7** *There exist connected interconnection  $\Gamma \in \mathbb{R}^{p \times p}$  and map  $Q : \mathbb{R} \rightarrow \overline{\mathcal{Q}}_n$  satisfying*

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \sigma_{\min} \left( \int_0^T Q_t dt \right) > 0$$

*such that solutions  $x_i(\cdot)$  of array (11) do not synchronize.*

**Proof.** As in the proof of the previous result, we will once again make use of projection matrices. Let  $\varepsilon_k = 2^{-k}$  for  $k = 1, 2, \dots$ . Then, let piecewise linear function  $\vartheta : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  be

$$\vartheta(t) := \vartheta(\tau_{k-1}) + t \sin \varepsilon_k \cos \varepsilon_k \quad \text{for } t \in [\tau_{k-1}, \tau_k)$$

where  $\tau_0 = 0$ ,  $\vartheta(0) = 0$ , and

$$\begin{aligned} \tau_k &= \tau_{k-1} + \frac{2\pi}{\sin \varepsilon_k \cos \varepsilon_k} \\ \vartheta(\tau_k) &= \lim_{t \rightarrow \tau_k^-} \vartheta(t) - \varepsilon_{k+1} \end{aligned}$$

for  $k = 1, 2, \dots$ . Now define  $Q : \mathbb{R}_{\geq 0} \rightarrow \overline{\mathcal{Q}}_2$  as

$$Q_t := \begin{bmatrix} \cos^2 \vartheta(t) & \sin \vartheta(t) \cos \vartheta(t) \\ \sin \vartheta(t) \cos \vartheta(t) & \sin^2 \vartheta(t) \end{bmatrix}$$

which is a projection matrix that projects onto the line spanned by the vector  $[\cos \vartheta(t) \ \sin \vartheta(t)]^T$ . Calculations yield

$$\frac{1}{\tau_k - \tau_{k-1}} \int_{\tau_{k-1}}^{\tau_k} Q_t dt = \frac{1}{2} I_2.$$

Observe that

$$\lim_{k \rightarrow \infty} \frac{\tau_{k-1}}{\tau_k} = \frac{1}{2} \tag{28}$$

and

$$\sigma_{\min} \left( \alpha I_2 + \int_{t_1}^{t_2} Q_t dt \right) = \alpha + \sigma_{\min} \left( \int_{t_1}^{t_2} Q_t dt \right)$$

for all  $\alpha \geq 0$  and  $t_2 \geq t_1 \geq 0$ . Given  $T \in (\tau_{k-1}, \tau_k]$  we can write

$$\begin{aligned} \frac{1}{T} \sigma_{\min} \left( \int_0^T Q_t dt \right) &= \frac{1}{T} \sigma_{\min} \left( \int_0^{\tau_{k-1}} Q_t dt + \int_{\tau_{k-1}}^T Q_t dt \right) \\ &= \frac{1}{T} \sigma_{\min} \left( \frac{\tau_{k-1}}{2} I_2 + \int_{\tau_{k-1}}^T Q_t dt \right) \\ &\geq \frac{\tau_{k-1}}{2T} \\ &\geq \frac{\tau_{k-1}}{2\tau_k} \end{aligned}$$

which yields by (28) that

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \sigma_{\min} \left( \int_0^T Q_t dt \right) \geq \frac{1}{4}.$$

Let us now consider (11) under the following connected interconnection

$$\Gamma := \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

Setting  $x_2(0) = 0$  we can write

$$\dot{x}_1 = -Q_t x_1. \quad (29)$$

Note that we need  $\lim_{t \rightarrow \infty} x_1(t) = 0$  for synchronization since  $x_2(\cdot) \equiv 0$ . In terms of polar coordinates, i.e.  $x_1 = [r \cos \theta \ r \sin \theta]^T$ , we can express (29) as

$$\dot{r} = -r \sin^2(\vartheta(t) - \theta - \pi/2) \quad (30a)$$

$$\dot{\theta} = \sin(\vartheta(t) - \theta - \pi/2) \cos(\vartheta(t) - \theta - \pi/2). \quad (30b)$$

Let us initialize  $x_1$  such that  $r > 0$  and  $\theta(0) = -\pi/2 - \varepsilon_1$ . We then observe that  $\dot{\theta}(t) = \dot{\vartheta}(t)$  for all  $t \geq 0$ . Eq. (30) simplifies to

$$\begin{aligned} \dot{r} &= -r \sin^2 \varepsilon_k \\ \dot{\theta} &= \sin \varepsilon_k \cos \varepsilon_k \end{aligned}$$

for  $t \in [\tau_{k-1}, \tau_k)$  and  $k = 1, 2, \dots$ . Thence

$$r(\tau_k) = r(\tau_{k-1}) e^{-(\tau_k - \tau_{k-1}) \sin^2 \varepsilon_k}$$

which yields

$$\begin{aligned} r(\tau_k) &= r(0) \exp \left( -2\pi \sum_{i=1}^k \tan \varepsilon_i \right) \\ &= r(0) \exp \left( -2\pi \sum_{i=1}^k \tan 2^{-i} \right) \\ &\geq r(0) \exp \left( -4\pi \sum_{i=1}^k 2^{-i} \right) \\ &\geq r(0) e^{-4\pi} \end{aligned}$$

for all  $k$ . Therefore  $\lim_{t \rightarrow \infty} x_1(t) \neq 0$ . ■

## 5 Observability grammian and synchronizability

Based on the results of the previous section, we are now ready to establish our theorems that are aimed to reveal the correlation between synchronizability and observability grammian. We begin with two definitions.

**Definition 4** For  $A : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$  and  $C : \mathbb{R} \rightarrow \mathbb{R}^{m \times n}$ , pair  $(C, A)$  is said to be asymptotically observable if the integrand of the observability grammian,  $t \mapsto \Phi_A^T(t, 0)C^T(t)C(t)\Phi_A(t, 0)$ , is sufficiently exciting.

The following definition is borrowed (with slight modification) from [6].

**Definition 5** For  $A : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$  and  $C : \mathbb{R} \rightarrow \mathbb{R}^{m \times n}$ , pair  $(C, A)$  is said to be uniformly observable if there exists a pair of positive real numbers  $(\varepsilon, T)$  such that

$$\sigma_{\min}(W_o(t, t+T)) \geq \varepsilon$$

for all  $t \geq 0$ .

**Remark 4** For  $A : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$  and  $C : \mathbb{R} \rightarrow \mathbb{R}^{m \times n}$  satisfying Assumption 1, asymptotic observability of pair  $(C, A)$  implies uniform observability of  $(C, A)$ . For a time-invariant pair, which need not satisfy Assumption 1; asymptotic observability, uniform observability, and the standard definition of observability (for time-invariant linear systems) are all equivalent.

The next result is our main theorem. It states that a time-varying pair  $(C, A)$  is synchronizable if it is asymptotically observable.

**Theorem 8** Let  $A : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$  and  $C : \mathbb{R} \rightarrow \mathbb{R}^{m \times n}$  satisfy Assumption 1. If pair  $(C, A)$  is asymptotically observable, then it is synchronizable. In particular, if we choose  $L : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$  as in (7), then for each  $\Gamma \in \mathcal{G}_{>0}$ , solutions  $x_i(\cdot)$  of array (5) with  $u_i = L(t)z_i$  synchronize to

$$\bar{x}(t) := (r^T \otimes \Phi_A(t, 0)) \begin{bmatrix} x_1(0) \\ \vdots \\ x_p(0) \end{bmatrix} \quad (31)$$

where  $r \in \mathbb{R}^p$  satisfies (3).

**Proof.** Let  $L$  be as in (7); then it is bounded by Assumption 1. Let  $\Gamma \in \mathbb{R}^{p \times p}$  be a connected interconnection and  $r \in \mathbb{R}^p$  satisfy (3). Consider array (5) with  $u_i = L(t)z_i$ . Define auxiliary variables  $\xi_i$  as

$$\xi_i(t) = \Phi_A(0, t)x_i(t). \quad (32)$$

Then, we can write

$$\dot{\xi}_i = Q_t \sum_{j \neq i} \hat{\gamma}_{ij}(\xi_j - \xi_i) \quad (33)$$

where  $Q_t := (\bar{a}\bar{c})^{-1}\Phi_A^T(t, 0)C^T(t)C(t)\Phi_A(t, 0)$ ,  $\hat{\gamma}_{ij} := \bar{a}\bar{c}\gamma_{ij}$ , and  $\bar{a}, \bar{c} \geq 1$  come from Assumption 1. Note that  $Q_t \in \overline{\mathcal{Q}}_n$  for all  $t \geq 0$ ,  $t \mapsto Q_t$  is sufficiently

exciting, and  $\widehat{\Gamma} := [\widehat{\gamma}_{ij}]$  is a connected interconnection satisfying  $r^T \widehat{\Gamma} = 0$ . We now invoke Theorem 4 on (33) to deduce that solutions  $\xi_i(\cdot)$  synchronize to

$$\bar{\xi}(t) \equiv (r^T \otimes I_n) \begin{bmatrix} x_1(0) \\ \vdots \\ x_p(0) \end{bmatrix} \quad (34)$$

Recall that  $\Phi_A$  is bounded. Hence, combining (34) and (32) yields (31).  $\blacksquare$

As noted in Remark 4, uniform observability is more restrictive a condition than asymptotic observability. However, it has a stronger outcome as stated by the following theorem.

**Theorem 9** *Let  $A : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$  and  $C : \mathbb{R} \rightarrow \mathbb{R}^{m \times n}$  satisfy Assumption 1. Then pair  $(C, A)$  is synchronizable if it is uniformly observable. In particular, if we choose  $L : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$  as in (7), then for each  $\Gamma \in \mathcal{G}_{>0}$ , solutions  $x_i(\cdot)$  of array (5) with  $u_i = L(t)z_i$  exponentially synchronize to*

$$\bar{x}(t) := (r^T \otimes \Phi_A(t, 0)) \begin{bmatrix} x_1(0) \\ \vdots \\ x_p(0) \end{bmatrix}$$

where  $r \in \mathbb{R}^p$  satisfies (3).

**Proof.** The demonstration flows in a way that is analogous to that of Theorem 8. This time, however, the result follows from Theorem 5.  $\blacksquare$

## 6 Discrete-time results

Our study on synchronization has hitherto been solely in continuous time. However, it is possible to extend the analysis to systems in discrete time without much difficulty. In fact, most of the continuous-time results have discrete-time counterparts under assumptions and definitions that are analogous to the ones we used for continuous-time systems. In this section, therefore, we focus on discrete-time time-varying linear systems and investigate the correlation between synchronizability and observability grammian in discrete time.

For a given interconnection  $\Lambda = [\lambda_{ij}] \in \mathbb{R}^{p \times p}$ , consider the array of  $p$  discrete-time linear systems, for  $k \in \mathbb{N}$ ,

$$x_i^+ = A(k)x_i + u_i \quad (35a)$$

$$y_i = C(k)x_i \quad (35b)$$

$$z_i = \sum_{j \neq i} \lambda_{ij}(y_j - y_i) \quad (35c)$$

where  $x_i \in \mathbb{R}^n$  is the state,  $x_i^+$  is the state at the next time instant,  $u_i \in \mathbb{R}^n$  is the input,  $y_i \in \mathbb{R}^m$  is the output, and  $z_i \in \mathbb{R}^m$  is the coupling of the  $i$ th system

for  $i = 1, 2, \dots, p$ . For each  $k \in \mathbb{N}$ , we have  $A(k) \in \mathbb{R}^{n \times n}$  and  $C(k) \in \mathbb{R}^{m \times n}$ . The solution of  $i$ th system at time  $k \in \mathbb{N}$  is denoted by  $x_i(k)$ . We denote by  $\Phi_A(\cdot, \cdot)$  the state transition matrix for  $A$ , i.e. for  $k > k_0$

$$\Phi_A(k, k_0) = A(k-1)A(k-2) \cdots A(k_0)$$

with  $\Phi_A(k_0, k_0) = I_n$ . We will let  $\Phi_A(k_0, k) = \Phi_A^{-1}(k, k_0)$  whenever the inverse exists. Observability grammian for pair  $(C, A)$  is given by

$$W_o(k_0, k) := \sum_{\ell=k_0}^{k-1} \Phi_A^T(\ell, k_0) C^T(\ell) C(\ell) \Phi_A(\ell, k_0)$$

for  $k, k_0 \in \mathbb{N}$ . Below we provide the discrete-time versions of Definition 1 and Assumption 1.

**Definition 6 (Synchronizability)** *Given functions  $A : \mathbb{N} \rightarrow \mathbb{R}^{n \times n}$  and  $C : \mathbb{N} \rightarrow \mathbb{R}^{m \times n}$ ; pair  $(C, A)$  is said to be synchronizable (with respect to  $\mathcal{G}_{>0}$ ) if there exists a bounded, time-varying linear feedback law  $L : \mathbb{N} \rightarrow \mathbb{R}^{n \times m}$  such that for each  $\Lambda \in \mathcal{G}_{>0}$ , solutions  $x_i(\cdot)$  of array (35) with  $u_i = L(k)z_i$  synchronize for all initial conditions.*

**Assumption 2 (Boundedness)** *For  $A : \mathbb{N} \rightarrow \mathbb{R}^{n \times n}$  and  $C : \mathbb{N} \rightarrow \mathbb{R}^{m \times n}$  following hold.*

- (a) *For each  $k \in \mathbb{N}$ ,  $A^{-1}(k)$  exists. There exists  $\bar{a} \geq 1$  such that  $|\Phi_A(k_1, k_2)| \leq \bar{a}$  for all  $k_1, k_2 \in \mathbb{N}$ .*
- (b) *There exists  $\bar{c} \geq 1$  such that  $|C(k)| \leq \bar{c}$  for all  $k \in \mathbb{N}$ .*

**Remark 5** *When  $A$  and  $C$  are constant matrices, Assumption 2(b) comes for free; and Assumption 2(a) becomes equivalent to that all eigenvalues of matrix  $A$  are with unity magnitude and none of them belongs to a Jordan block with size two or greater.*

## 6.1 Synchronization under bounded SPSD matrix

This subsection will emulate Section 4, where we studied the stability and synchronization properties of array (11). For a given interconnection  $\Lambda = [\lambda_{ij}] \in \mathbb{R}^{p \times p}$ , let an array of  $p$  systems be

$$x_i^+ = x_i + Q_k \sum_{j \neq i} \lambda_{ij} (x_j - x_i) \quad (36)$$

where  $x_i \in \mathbb{R}^n$  and  $Q_k \in \overline{\mathcal{Q}}_n$  for all  $k \in \mathbb{N}$ . We consider (36) as the discrete-time analogue of (11). Let us stack individual vectors  $x_i$  into  $\mathbf{x} = [x_1^T \ x_2^T \ \dots \ x_p^T]^T$ . Then we obtain from (36)

$$\mathbf{x}^+ = (I_{np} + (\Lambda - I_p) \otimes Q_k) \mathbf{x} \quad (37)$$

which makes the analogue of system (12).

**Remark 6** Recall that to establish stability of (continuous-time) array (11) it sufficed that  $Q_t$  is SPSPD for each  $t$ . (See Theorem 1.) That is, boundedness was not required. Later, when we established synchronization in Theorem 4, we needed solely that  $Q : \mathbb{R} \rightarrow \mathcal{Q}_n$  is bounded. (See Remark 2.) The story has to be a little bit different in discrete-time. Note that in (36) we stipulated that  $Q_k$  (the discrete-time counterpart of  $Q_t$ ) be in  $\overline{\mathcal{Q}}_n$ . Even to be able to establish stability, let alone synchronization, we will need  $|Q_k| \leq 1$ , a more restrictive condition than boundedness. Clearly, this has to do with the fact that for discrete-time systems the magnitude of the righthand side is important for stability; whereas in continuous time, what matters (for stability) is only the direction of the righthand side.

**Lemma 3** Let interconnection  $\Lambda \in \mathbb{R}^{p \times p}$  be connected and  $r \in \mathbb{R}^p$  satisfy (4). Then, there exists symmetric positive definite matrix  $\Omega \in \mathbb{R}^{p \times p}$  such that

$$(\Lambda - \mathbf{1}r^T)^T \Omega (\Lambda - \mathbf{1}r^T) - \Omega = -I_p. \quad (38)$$

**Proof.** We first observe that  $(\Lambda - \mathbf{1}r^T)^k = \Lambda^k - \mathbf{1}r^T$ . Then we can write  $\lim_{k \rightarrow \infty} \Lambda^k - \mathbf{1}r^T = 0$ , which implies that matrix  $[\Lambda - \mathbf{1}r^T]$  is Schur, i.e. all of its eigenvalues are strictly within unit circle. Therefore, discrete-time Lyapunov equation (38) admits a symmetric positive definite solution  $\Omega$ . ■

**Lemma 4** Let interconnection  $\Lambda \in \mathbb{R}^{p \times p}$  be connected,  $r \in \mathbb{R}^p$  satisfy (4), and symmetric positive definite matrix  $\Omega \in \mathbb{R}^{p \times p}$  satisfy (38). Define  $V : \mathbb{R}^{np} \rightarrow \mathbb{R}_{\geq 0}$  as  $V(\mathbf{x}) := \mathbf{x}^T (\Omega \otimes I_n) \mathbf{x}$ . Then, for all  $Q : \mathbb{N} \rightarrow \overline{\mathcal{Q}}_n$  and all  $k \in \mathbb{N}$ , solution of system (37) satisfies

$$V(\mathbf{x}(k+1) - \bar{\mathbf{x}}) - V(\mathbf{x}(k) - \bar{\mathbf{x}}) \leq -(\mathbf{x}(k) - \bar{\mathbf{x}})^T (I_p \otimes Q_k^2) (\mathbf{x}(k) - \bar{\mathbf{x}}) \quad (39)$$

where  $\bar{\mathbf{x}} := (\mathbf{1}r^T \otimes I_n) \mathbf{x}(0)$ .

**Proof.** Observe that  $(\mathbf{1}r^T \otimes I_n) \mathbf{x}(k+1) = \mathbf{x}(k)$  whence  $(\mathbf{1}r^T \otimes I_n) \mathbf{x}(k) = \bar{\mathbf{x}}$  for all  $k \in \mathbb{N}$ . Let  $\xi := \mathbf{x} - \bar{\mathbf{x}}$  and  $\Lambda_o := \Lambda - \mathbf{1}r^T$ . Then we have  $\xi^+ = (I_{np} + (\Lambda_o - I_p) \otimes Q_k) \xi$ . We can write

$$\begin{aligned} V(\xi^+) - V(\xi) &= \xi^T \left( (I_{np} + (\Lambda_o - I_p) \otimes Q_k)^T (\Omega \otimes I_n) (I_{np} + (\Lambda_o - I_p) \otimes Q_k) - \Omega \otimes I_n \right) \xi \\ &= \xi^T \left( ((\Lambda_o - I_p)^T \Omega + \Omega (\Lambda_o - I_p)) \otimes (Q_k - Q_k^2) + (\Lambda_o^T \Omega \Lambda_o - \Omega) \otimes Q_k^2 \right) \xi. \end{aligned}$$

Note that  $Q_k - Q_k^2 \geq 0$  since  $|Q_k| \leq 1$  and that  $(\Lambda_o - I_p)^T \Omega + \Omega (\Lambda_o - I_p) < 0$  since  $\Lambda_o^T \Omega \Lambda_o - \Omega < 0$  (this is almost immediate when we recall that the sublevel sets of quadratic Lyapunov functions are convex surfaces.) Therefore

$$V(\xi^+) - V(\xi) \leq -\xi^T (I_p \otimes Q_k^2) \xi.$$

Hence the result. ■

**Remark 7** Were  $Q_k$  a projection matrix, then inequality (39) could be replaced by the below equality

$$V(\mathbf{x}(k+1) - \bar{\mathbf{x}}) - V(\mathbf{x}(k) - \bar{\mathbf{x}}) = -(\mathbf{x}(k) - \bar{\mathbf{x}})^T (I_p \otimes Q_k) (\mathbf{x}(k) - \bar{\mathbf{x}}).$$

**Theorem 10** Given interconnection  $\Lambda \in \mathbb{R}^{p \times p}$ , there exists  $\alpha > 0$  such that for all  $Q : \mathbb{N} \rightarrow \overline{\mathcal{Q}}_n$ , solution of system (37) satisfies

$$|\mathbf{x}(k)| \leq \alpha |\mathbf{x}(0)|$$

for all  $k \in \mathbb{N}$ .

**Proof.** Demonstration uses Lemma 4 and flows similar to that of Theorem 1. ■

**Fact 3** For map  $Q : \mathbb{N} \rightarrow \overline{\mathcal{Q}}_n$ , real number  $\varepsilon > 0$ , and integer  $N \geq 1$  we have

$$\sigma_{\min} \left( \sum_{k=0}^{N-1} Q_k \right) \geq \varepsilon \implies \sigma_{\min} \left( \sum_{k=0}^{N-1} Q_k^2 \right) \geq \frac{\varepsilon^2}{N^2 n^2}.$$

**Proof.** Let us be given any  $v \in \mathbb{R}^n$  with  $v^T v = 1$ . Let  $\sigma_{\min} \left( \sum_{k=0}^{N-1} Q_k \right) \geq \varepsilon > 0$ . We can write

$$v^T Q_0 v + v^T Q_1 v + \dots + v^T Q_{N-1} v \geq \varepsilon$$

which implies that there exists  $k^* \in \{0, 1, \dots, N-1\}$  such that  $v^T Q_{k^*} v \geq \varepsilon/N$ . Since  $Q_{k^*}$  is an SPSD matrix there exists an orthogonal matrix  $R \in \mathbb{R}^{n \times n}$  and a diagonal matrix  $D \in \mathbb{R}^{n \times n}$  with entries  $d_i \in [0, 1]$  for  $i = 1, 2, \dots, n$  such that  $Q_{k^*} = R^T D R$ . Also note that  $Q_{k^*}^2 = R^T D^2 R$ . Let  $[w_1 \ w_2 \ \dots \ w_n]^T = w := Rv$ . Note that  $w^T w = 1$  for  $R$  is orthogonal. We can therefore write

$$\sum_{i=1}^n d_i w_i^2 \geq \frac{\varepsilon}{N}$$

which implies that there exists  $i^* \in \{1, 2, \dots, n\}$  such that  $d_{i^*} w_{i^*}^2 \geq \varepsilon/(Nn)$ . Since  $w_{i^*}^2 \leq 1$ , we can write

$$\begin{aligned} \frac{\varepsilon^2}{N^2 n^2} &\leq d_{i^*}^2 w_{i^*}^4 \\ &\leq d_{i^*}^2 w_{i^*}^2 \\ &\leq w^T D^2 w \\ &= v^T Q_{k^*}^2 v \\ &\leq v^T \left( \sum_{k=0}^{N-1} Q_k^2 \right) v \end{aligned}$$

whence the result follows, for  $v$  was arbitrary. ■

**Theorem 11** Let  $\varepsilon > 0$  be a real number and  $N \geq 1$  an integer. Define

$$\delta(\varepsilon, N) := \frac{\varepsilon^4}{16N^8n^4}. \quad (40)$$

Let interconnection  $\Lambda \in \mathbb{R}^{p \times p}$  be connected,  $r \in \mathbb{R}^p$  and symmetric positive definite matrix  $\Omega \in \mathbb{R}^{p \times p}$ , respectively, satisfy (4) and (38). Define

$$\rho(\Lambda) := \sigma_{\max}(\Omega) \max\{1, |\Lambda - I_p|^2\}. \quad (41)$$

Let  $V(\mathbf{x}) := \mathbf{x}^T(\Omega \otimes I_n)\mathbf{x}$  for  $\mathbf{x} \in \mathbb{R}^{np}$ . Then, for all  $Q : \mathbb{N} \rightarrow \overline{\mathcal{Q}}_n$ , the below inequality

$$\sigma_{\min} \left( \sum_{k=0}^{N-1} Q_k \right) \geq \varepsilon \quad (42)$$

implies that solution of system (37) satisfies

$$V(\mathbf{x}(N) - \bar{\mathbf{x}}) \leq \left( 1 - \frac{\delta(\varepsilon, N)}{\rho(\Lambda)} \right) V(\mathbf{x}(0) - \bar{\mathbf{x}})$$

where  $\bar{\mathbf{x}} := (\mathbf{1}r^T \otimes I_n)\mathbf{x}(0)$ .

**Proof.** Given pair  $(\varepsilon, T)$  let  $\omega := \varepsilon^2/(4N^3n^2)$ . Consider system (37). Let us introduce

$$\xi(k) := \mathbf{x}(k) - \bar{\mathbf{x}}.$$

By Lemma 4 we have

$$V(\xi(k+1)) - V(\xi(k)) \leq -\xi^T(k)(I_p \otimes Q_k^2)\xi(k). \quad (43)$$

Also,  $\xi$  can be shown to satisfy

$$\xi^+ = (I_{np} + (\Lambda - I_p) \otimes Q_k)\xi. \quad (44)$$

Let  $Q : \mathbb{N} \rightarrow \overline{\mathcal{Q}}_n$  satisfy (42). Then, regarding the evolution of  $\xi(\cdot)$ , one of the two following cases must be.

*Case 1:*  $|\xi(k) - \xi(0)| \leq \omega|\xi(0)|$  for all  $k \in \{1, 2, \dots, N-1\}$ . Let  $b(k) := \xi(k) - \xi(0)$  and recall that  $|Q_k| \leq 1$ . Let  $\mathbf{Q}_k := (I_p \otimes Q_k)$ . Note that then we have  $|\mathbf{Q}_k| \leq 1$  as well as, by Fact 3,

$$\begin{aligned} \sigma_{\min} \left( \sum_{k=0}^{N-1} \mathbf{Q}_k^2 \right) &= \sigma_{\min} \left( I_p \otimes \sum_{k=0}^{N-1} Q_k^2 \right) \\ &\geq \frac{\varepsilon^2}{N^2n^2}. \end{aligned}$$

From (43) we can write

$$\begin{aligned}
V(\xi(N)) &\leq V(\xi(0)) - \sum_{k=0}^{N-1} \xi^T(k) \mathbf{Q}_k^2 \xi(k) \\
&= V(\xi(0)) - \sum_{k=0}^{N-1} (\xi(0) + b(k))^T \mathbf{Q}_k^2 (\xi(0) + b(k)) \\
&= V(\xi(0)) - \xi^T(0) \left( \sum_{k=0}^{N-1} \mathbf{Q}_k^2 \right) \xi(0) - 2 \sum_{k=0}^{N-1} b^T(k) \mathbf{Q}_k^2 \xi(0) - \sum_{k=0}^{N-1} b^T(k) \mathbf{Q}_k^2 b(k) \\
&\leq V(\xi(0)) - \frac{\varepsilon^2}{N^2 n^2} |\xi(0)|^2 + 2\omega N |\xi(0)|^2 \\
&= V(\xi(0)) - \frac{\varepsilon^2}{2N^2 n^2} |\xi(0)|^2 \\
&\leq \left( 1 - \frac{\varepsilon^2}{2N^2 n^2 \sigma_{\max}(\Omega)} \right) V(\xi(0)). \tag{45}
\end{aligned}$$

*Case 2:*  $|\xi(\bar{k}) - \xi(0)| \geq \omega |\xi(0)|$  for some  $\bar{k} \in \{1, 2, \dots, N-1\}$ . We can by (44) write

$$\begin{aligned}
\sum_{k=0}^{\bar{k}} |\mathbf{Q}_k \xi(k)| &= |\Lambda - I_p|^{-1} \sum_{k=0}^{\bar{k}} |(\Lambda - I_p) \otimes I_n| |\mathbf{Q}_k \xi(k)| \\
&\geq |\Lambda - I_p|^{-1} \sum_{k=0}^{\bar{k}} |((\Lambda - I_p) \otimes Q_k) \xi(k)| \\
&= |\Lambda - I_p|^{-1} \sum_{k=0}^{\bar{k}} |\xi(k+1) - \xi(k)| \\
&\geq |\Lambda - I_p|^{-1} \left| \sum_{k=0}^{\bar{k}-1} \xi(k+1) - \xi(k) \right| \\
&= |\Lambda - I_p|^{-1} |\xi(\bar{k}) - \xi(0)| \\
&\geq |\Lambda - I_p|^{-1} \omega |\xi(0)|. \tag{46}
\end{aligned}$$

Eq. (46) implies that there exists  $k \in \{0, 1, \dots, \bar{k}\}$  such that  $|\mathbf{Q}_k \xi(k)| \geq |\Lambda - I_p|^{-1} \omega |\xi(0)|/N$  which implies

$$\sum_{k=0}^{N-1} |\mathbf{Q}_k \xi(k)|^2 \geq |\Lambda - I_p|^{-2} \omega^2 |\xi(0)|^2 / N^2.$$

Then, by (43) we can write

$$\begin{aligned}
V(\xi(N)) &= V(\xi(0)) - \sum_{k=0}^{N-1} \xi^T(k) \mathbf{Q}_k^2 \xi(k) \\
&\leq V(\xi(0)) - \frac{\omega^2}{|\Lambda - I_p|^2 N^2} |\xi(0)|^2 \\
&\leq \left( 1 - \frac{\varepsilon^4}{16N^8 n^4 |\Lambda - I_p|^2 \sigma_{\max}(\Omega)} \right) V(\xi(0))
\end{aligned} \tag{47}$$

The result follows by (40), (41), (45), and (47).  $\blacksquare$

Theorem 11 suggests the following definition.

**Definition 7** *Function  $Q : \mathbb{N} \rightarrow \mathcal{Q}_n$  is said to be sufficiently exciting if there exists a sequence of pairs of positive real numbers  $(\varepsilon_i, N_i)_{i=1}^\infty$  satisfying*

$$\sigma_{\min} \left( \sum_{k=k_i}^{k_i+N_i-1} Q_k \right) \geq \varepsilon_i \tag{48}$$

for  $k_i = \sum_{j=1}^{i-1} N_j$  with  $k_1 = 0$ , and

$$\lim_{N \rightarrow \infty} \sum_{i=1}^N \delta(\varepsilon_i, N_i) = \infty \tag{49}$$

where  $\delta(\cdot, \cdot)$  is as defined in (40).

The following result is the discrete-time analogue of Theorem 4. The proof would have been similar to that of Theorem 4, had it not been absent from the paper.

**Theorem 12** *Let  $Q : \mathbb{N} \rightarrow \overline{\mathcal{Q}}_n$  be sufficiently exciting. Then, for all connected interconnection  $\Lambda \in \mathbb{R}^{p \times p}$ , solutions  $x_i(\cdot)$  of array (36) synchronize to*

$$\bar{x}(k) \equiv (r^T \otimes I_n) \mathbf{x}(0)$$

where  $r \in \mathbb{R}^p$  satisfies (4).

Notion of persistence of excitation carries readily to discrete time. See the below definition.

**Definition 8** *Map  $Q : \mathbb{N} \rightarrow \mathcal{Q}_n$  is said to be persistently exciting if there exists a pair  $(\varepsilon, N)$ ,  $\varepsilon > 0$  and  $N \in \mathbb{N}_{\geq 1}$ , such that*

$$\sigma_{\min} \left( \sum_{k=k_0}^{k_0+N-1} Q_k \right) \geq \varepsilon \tag{50}$$

for all  $k_0 \in \mathbb{N}$ .

The following theorem is the discrete-time analogue of Theorem 5. We omit the proof.

**Theorem 13** *Let interconnection  $\Lambda \in \mathbb{R}^{p \times p}$  be connected and function  $Q : \mathbb{N} \rightarrow \overline{\mathcal{Q}}_n$  persistently exciting. Then solutions  $x_i(\cdot)$  of array (36) exponentially synchronize to*

$$\bar{x}(k) \equiv (r^T \otimes I_n)\mathbf{x}(0)$$

where  $r \in \mathbb{R}^p$  satisfies (4).

## 6.2 Negative results in discrete time

Negative results generated in Subsection 4.3 are not peculiar to continuous-time arrays. Counterexamples similar to the ones constructed in the proofs of Theorem 6 and Theorem 7 can be obtained in discrete time. We thus have the following two theorems.

**Theorem 14** *There exist maps  $\Lambda : \mathbb{N} \rightarrow \mathbb{R}^{p \times p}$ , where  $\Lambda_k$  is an interconnection for each  $k$ , and  $Q : \mathbb{N} \rightarrow \overline{\mathcal{Q}}_n$  such that system  $\mathbf{x}^+ = (I_{np} + (\Lambda_k - I_p) \otimes Q_k)\mathbf{x}$  has an unbounded solution.*

**Theorem 15** *There exist connected interconnection  $\Lambda \in \mathbb{R}^{p \times p}$  and map  $Q : \mathbb{N} \rightarrow \overline{\mathcal{Q}}_n$  satisfying*

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \sigma_{\min} \left( \sum_{k=0}^N Q_k \right) > 0$$

such that solutions  $x_i(\cdot)$  of array (36) do not synchronize.

## 6.3 Observability grammian and synchronizability in discrete time

As is the case with continuous-time arrays, there is a close relation between the observability grammian and synchronizability in discrete time. In this subsection we will provide definitions and theorems through which we formalize that relation.

**Definition 9** *For  $A : \mathbb{N} \rightarrow \mathbb{R}^{n \times n}$  and  $C : \mathbb{N} \rightarrow \mathbb{R}^{m \times n}$ , pair  $(C, A)$  is said to be asymptotically observable if the summand of the observability grammian,  $k \mapsto \Phi_A^T(k, 0)C^T(k)C(k)\Phi_A(k, 0)$ , is sufficiently exciting.*

**Definition 10** *For  $A : \mathbb{N} \rightarrow \mathbb{R}^{n \times n}$  and  $C : \mathbb{N} \rightarrow \mathbb{R}^{m \times n}$ , pair  $(C, A)$  is said to be uniformly observable if there exists a pair  $(\varepsilon, N)$ ,  $\varepsilon > 0$  and  $N \in \mathbb{N}_{\geq 1}$ , such that*

$$\sigma_{\min}(W_o(k, k + N)) \geq \varepsilon$$

for all  $k \in \mathbb{N}$ .

The below result follows from Theorem 12.

**Theorem 16** *Let  $A : \mathbb{N} \rightarrow \mathbb{R}^{n \times n}$  and  $C : \mathbb{N} \rightarrow \mathbb{R}^{m \times n}$  satisfy Assumption 2 (with constants  $\bar{a}$  and  $\bar{c}$ .) If pair  $(C, A)$  is asymptotically observable, then it is synchronizable. In particular, if we choose  $L : \mathbb{N} \rightarrow \mathbb{R}^{n \times n}$  as*

$$L(k) := (\bar{a}\bar{c})^{-1}\Phi_A(k+1, 0)\Phi_A^T(k, 0)C^T(k) \quad (51)$$

*then for each  $\Lambda \in \mathcal{G}_{>0}$  solutions  $x_i(\cdot)$  of array (35) synchronize to*

$$\bar{x}(k) := (r^T \otimes \Phi_A(k, 0)) \begin{bmatrix} x_1(0) \\ \vdots \\ x_p(0) \end{bmatrix}$$

*where  $r \in \mathbb{R}^p$  satisfy (4).*

Theorem 13 yields the following result.

**Theorem 17** *Let  $A : \mathbb{N} \rightarrow \mathbb{R}^{n \times n}$  and  $C : \mathbb{N} \rightarrow \mathbb{R}^{m \times n}$  satisfy Assumption 2 (with constants  $\bar{a}$  and  $\bar{c}$ .) Then pair  $(C, A)$  is synchronizable if it is uniformly observable. In particular, if we choose  $L : \mathbb{N} \rightarrow \mathbb{R}^{n \times n}$  as in (51) then for each  $\Lambda \in \mathcal{G}_{>0}$  solutions  $x_i(\cdot)$  of array (35) exponentially synchronize to*

$$\bar{x}(k) := (r^T \otimes \Phi_A(k, 0)) \begin{bmatrix} x_1(0) \\ \vdots \\ x_p(0) \end{bmatrix}$$

*where  $r \in \mathbb{R}^p$  satisfy (4).*

## 7 Conclusion

We studied synchronization of stable, linear time-varying systems that are coupled via their outputs. We provided sufficient conditions on observability gramian for the existence of a bounded linear feedback law under which the systems synchronize for all fixed connected interconnections. Related to the main problem, we also studied an array of coupled integrators with identical time-varying output matrices that are symmetric positive semi-definite. We showed, via Lyapunov arguments that, the trajectories of this array stay bounded. Moreover, if the interconnection is connected and output matrix satisfies some observability condition, then the systems were shown to reach consensus.

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